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Constraints, causality and Lorentz invariance

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Abstract. Simple theories of interacting classical relativistic massive spin-1 and spin-$\frac{3}{2}$ fields are studied. In these examples, it is found that if a theory is acausal in the sense of Velo and Zwanziger, it is also not Lorentz covariant. The breakdown of Lorentz covariance is seen to manifest itself in the constraints, which must be satisfied by the interacting fields, not determining all the dependent fields components simultaneously in all inertial frames of reference. Some remarks on the present state of massive spin-$\frac{3}{2}$ theories are made.

1. Introduction

Ever since the discovery of relativistic wave equations which describe a free particle of unique mass and spin, the attempts to construct interacting higher-spin theories, by the usual method of inserting apparently Lorentz-covariant source terms into the free wave equations, have been fraught with difficulties at both the classical and quantum levels. The main body of the present paper will only be concerned with such difficulties at the classical level. Thus, from here on, all considerations will be understood to be classical, unless otherwise stated.

The most prominent difficulty with interacting higher-spin wave equations, to come to light in recent years, is the possibility that, even when source terms are constructed in an apparently Lorentz-covariant manner, the wave equations may possess solutions which are acausal, in the sense that they can propagate across the light cone (Velo and Zwanziger 1969a, b, 1971).

Another difficulty has been noted and discussed by Federbush (1961), Velo and Zwanziger (1969b) and Velo (1972). They considered the interaction of a massive, charged, spin-2 particle with an external electromagnetic field, and it was found that the presence of magnetic dipole interactions leads, in general, to there being six degrees of freedom, rather than the five required for the description of a massive spin-2 particle. In addition, Nath *et al* (1971) and Hagen (1971) have noted that there are certain simple forms of the field-theoretic $\pi N \Delta$ vertex, which lead to a loss of constraints, for the massive spin-$\frac{3}{2}$ particle, which is analogous to the loss of constraints in the above-mentioned spin-2 case. Note that such losses of constraints are Lorentz-covariant phenomena, in the sense that they occur independently of the frame of reference.

In the present paper, relativistic field theories will be studied, which are acausal in the sense of Velo and Zwanziger (1969a, b, 1971), and it will be seen that, in the examples considered, such acausality is accompanied by a loss of constraints. In contrast with the losses of constraints mentioned above for spins $\frac{3}{2}$ and 2, the losses of constraints found are not Lorentz-covariant phenomena. The correct number of constraints is present in some frames of reference, whilst constraints are lost in others. Thus the theories to be presented here, in addition to being acausal, are not Lorentz covariant.

The plan of the paper is as follows. In § 2, the connections between causality, constraints and Lorentz invariance are illustrated by a simple example of an interacting massive spin-1 particle. Section 3 contains an outline of the parallel discussion of interactions bilinear in the Dirac field and a massive spin-$\frac{3}{2}$ field. Finally, § 4 is devoted to a discussion of the results of §§ 2 and 3, and their implications.

2. Spin 1

By way of illustration, a discussion of the constraint equations, which are satisfied by a massive spin-1 field, is given for the following simple example, which is known to be acausal at the classical level (Velo and Zwanziger 1969b) and which, when quantized, gives a theory which is not Lorentz covariant (Jenkins 1973a):

$$L_\mu \equiv (\partial^2+m^2)V_\mu(x)-\partial_\mu\partial^\lambda V_\lambda(x)-T_{\mu\lambda}(x)V^\lambda(x) = 0 \tag{1}$$

where $T_{\mu\nu}(x)$ is a symmetric, second-rank tensor, external potential, which is not proportional to $g_{\mu\nu}$.

As is the case in all parity-invariant, relativistic, higher-spin field theories, the field equations (1) are degenerate, in that they include primary constraints, namely

$$L_0 = 0. \tag{2}$$

There are, here, also secondary constraints given by

$$\partial^\mu L_\mu \equiv m^2\partial^\mu V_\mu(x)-\partial^\mu(T_{\mu\lambda}(x)V^\lambda(x)) = 0. \tag{3}$$

In principle (2) and (3) should be solvable for $V_0(x)$ and $\partial_0 V_0(x)$ in terms of $V_i(x)$, $\partial_\mu V_i(x)$ and $T_{\mu\nu}(x)$, in order that the vector field should have the three independent components required for the description of a massive spin-1 particle. This point will be returned to later.

Because of the field equations (1) being degenerate, they do not form a hyperbolic system, and hence the wave nature of the solutions is not manifest. However, provided that $T_{\mu\nu}(x)$ is appropriately restricted (Velo and Zwanziger 1969b), the wave nature of the solutions may be made manifest by using the secondary constraint (3), in (1), to derive the following hyperbolic system of equations for $V_\mu(x)$:

$$L_\mu - m^{-2}\partial_\mu\partial^\lambda L_\lambda = 0. \tag{4}$$

The equivalence of the hyperbolic system (4) to the original field equations (1) is assured when (4) is supplemented by the constraints (2) and (3), assumed valid only on the 'initial' space-like surface x^0 = constant, it being then possible to show that (2) and (3) are satisfied at all later times (Velo and Zwanziger 1969b). Finally the Cauchy problem may be set up for (4), with 'initial' data given on the space-like surface x^0 = constant and satisfying the constraints (2) and (3).

It should be noted that the above procedure is not manifestly Lorentz covariant. Thus, in order to check whether or not a breakdown of Lorentz invariance has been obscured by this (usual) approach, the latter is now translated into a manifestly Lorentz-covariant form.

In the manifestly Lorentz-covariant approach to the above, the 'initial' space-like surface x^0 = constant is replaced by an arbitrary space-like surface σ with unit (time-like) normal n_μ. The Cauchy problem may now be set up for (4), with 'initial' data given on the space-like surface σ and satisfying the constraints

$$n^\mu L_\mu = 0 \tag{2a}$$

and (3). However, these constraints should now be solvable for $n^\mu V_\mu(x)$ and $n^\lambda \partial_\lambda n^\rho V_\rho(x)$ in terms of $V_{t\mu}(x)$, $\partial_\mu V_{t\nu}(x)$ and $T_{\mu\nu}(x)$, where

$$V_{t\mu}(x) = V_\mu(x) - n_\mu n^\rho V_\rho(x) \tag{5}$$

and satisfies

$$n^\mu V_{t\mu}(x) = 0. \tag{6}$$

It is readily seen by inspection that (2*a*) is, in principle, solvable for $n^\mu V_\mu(x)$ in the above manner, whilst (6) evidently presents no problems. On the other hand, note that (3) may be rewritten in the form

$$(m^2 - n^\mu n^\nu T_{\mu\nu}(x)) n^\lambda \partial_\lambda n^\rho V_\rho(x) = f(n^\lambda V_\lambda(x), V_{t\lambda}(x), \partial_\lambda V_{t\rho}(x), T_{\mu\nu}(x)) \tag{7}$$

where the form of the right-hand side of (7) is immaterial for the following discussion, and where after solving (2*a*) for it, $n^\lambda V_\lambda(x)$ may be eliminated from the right-hand side.

Inspection of (7) reveals that it is solvable for $n^\lambda \partial_\lambda n^\rho V_\rho(x)$, on the 'initial' space-like surface σ, if and only if $m^2 - n^\mu n^\nu T_{\mu\nu}(x)$ does not vanish for n_μ the unit normal to σ. Note however that, since $n^2 = 1$, $m^2 - n^\mu n^\nu T_{\mu\nu}(x)$ is just proportional to the characteristic determinant of the system of equations (4), and that $T_{\mu\nu}(x)$ can be such that there exists a real $n_\mu(n^2 = 1)$ such that $m^2 - n^\mu n^\nu T_{\mu\nu}(x)$ vanishes (Velo and Zwanziger 1969b). Thus, for such $T_{\mu\nu}(x)$, there is a space-like surface σ, with unit normal n_μ, and obtainable from x^0 = constant by a Lorentz transformation, on which (7) is not solvable for $n^\lambda \partial_\lambda n^\rho V_\rho(x)$.

Thus, for such $T_{\mu\nu}(x)$, the Cauchy problem for (4) with the constraints (2*a*) and (3) satisfied on the 'initial' space-like surface does not take the same form in all inertial frames of reference, in that the form of the constraint (3) can vary with the frame of reference. This variation is just a manifestation of the theory's not being Lorentz covariant.

3. Spin $\frac{3}{2}$

Interactions, involving the Dirac field, a massive spin-$\frac{3}{2}$ field and external potentials, and which are bilinear in the Dirac and spin-$\frac{3}{2}$ fields, are considered. It might have been hoped that such theories may be less pathological than theories quadratic in a massive spin-$\frac{3}{2}$ field, however, as will be seen below, there are difficulties with Lorentz covariance in all but a trivial class of such theories. Interactions with external potentials, rather than other fields, are considered so as to facilitate the extraction of the essence of the difficulties, arising from the constraints which must be satisfied by the spin-$\frac{3}{2}$ field. These difficulties would not be removed if the external potentials were replaced by interacting fields.

The field equations to be considered are given by

$$\Lambda_{\mu\nu}(\partial)\psi^\nu(x) = A_\mu(x)\psi(x) \equiv J_\mu \tag{8}$$

$$(\mathrm{i}\not{\partial} - m)\psi(x) = \bar{A}^\lambda(x)\psi_\lambda(x) \equiv K \tag{9}$$

where $A_\mu(x)$ is a covariant 4×4 matrix potential,

$$\Lambda_{\mu\nu}(\partial) \equiv -(\mathrm{i}\partial\!\!\!/ - M)g_{\mu\nu} + \mathrm{i}(\gamma_\mu\partial_\nu + \gamma_\nu\partial_\mu) - \mathrm{i}\gamma_\mu\partial\!\!\!/\gamma_\nu - M\gamma_\mu\gamma_\nu \tag{10}$$

and

$$\bar{A}_\mu = \gamma_0 A_\mu^\dagger \gamma_0. \tag{11}$$

The primary constraints satisfied by the spin-$\frac{3}{2}$ field are

$$n^\mu \Lambda_{\mu\nu}(\partial)\psi^\nu(x) = n^\mu J_\mu \tag{12}$$

whilst the secondary constraints are obtained from

$$\frac{2\mathrm{i}}{M}\partial^\mu J_\mu - 3M\gamma^\mu\psi_\mu(x) = \gamma^\mu J_\mu \tag{13}$$

by eliminating the normal derivative, $n^\mu\partial_\mu\psi(x)$, using (9). It should now be noted that the secondary constraints are used in the field equations (8) to generate the equivalent hyperbolic system, and that, if there is to be any hope of the theory being Lorentz covariant, the latter must have no n dependence. To ensure this, (13) must remain n independent even after the elimination of the normal derivative terms. This can only occur if the restriction

$$A_\mu(x) = A(x)\gamma_\mu,$$

where $A(x)$ is a 4×4 matrix, is made, and then, after use of (9), (13) becomes

$$\frac{2\mathrm{i}}{M}\partial^\mu A(x)\gamma_\mu\psi(x) + \frac{2}{M}A(x)(m\psi(x) + \gamma^\mu\bar{A}(x)\psi_\mu(x)) - 3M\gamma^\mu\psi_\mu(x) = \gamma^\mu J_\mu. \tag{14}$$

Since $n^\mu\Lambda_{\mu\nu}(\partial)n^\nu$ and $n^\mu n^\nu\Lambda_{\mu\nu}(\partial)$ both vanish, (12) is, as expected, a constraint between just the components $\psi_{t\mu}(x)$ of the spin-$\frac{3}{2}$ field, where the subscript 't' has the meaning defined by (5); and it presents no difficulties. On the other hand, (14) should be solvable for $n^\mu\psi_\mu(x)$ in terms of the independent field components and the external potentials. It is readily seen that (14) is so solvable if and only if

$$\left|1 - \frac{2}{3M^2}A(x)n\!\!\!/\bar{A}(x)n\!\!\!/\right| \neq 0 \tag{15}$$

where 1 is the unit 4×4 matrix. Thus, analogously to the case of spin 1 discussed in § 2, the Lorentz covariance of the present theory is assured only if the above determinant has no real unit time-like root. The only form for $A(x)$, which guarantees that this is so is

$$A(x) = \alpha(x)1 + \beta(x)\gamma_5, \tag{16}$$

where $\alpha(x)$ and $\beta(x)$ are Lorentz-scalar densities, and then the above determinant reduces to

$$\left|\left(1 - \frac{2}{3M^2}(|\alpha(x)|^2 + |\beta(x)|^2)\right)1 - \frac{2}{3M^2}(\alpha^*(x)\beta(x) - \alpha(x)\beta^*(x))\gamma_5\right|. \tag{17}$$

Singh (1973) has studied the causal nature of the propagation of Dirac and massive spin-$\frac{3}{2}$ fields interacting bilinearly in an external potential. Although the form of the potential, considered by him, is different from the above, a treatment of the above, which is exactly parallel to his work, leads to the conclusion that the above theory is acausal unless $A(x)$ is the form (16). Because the calculations follow exactly Singh's work, the details are omitted.

Finally, it should be noted that, when $A(x)$ has the form (16), (8), (13) and their conjugates may be used to eliminate $\psi_\mu(x)$ and $\bar{\psi}_\mu(x)$ from the lagrangian which leads to (8) and (9), thus demonstrating that the theory then effectively just involves an interaction between the Dirac field and the external potential.

4. Discussion

The main result of §§ 2 and 3 is that the acausal classical relativistic field theories (Velo and Zwanziger 1969a, b, 1971), which are considered therein, are also not Lorentz covariant. This is so despite the field equations, in each case, being derivable from a lagrangian, in which the interaction part is constructed from the fields appearing in the free part (and from external potentials), in such a manner that, were the Lorentz-transformation properties of the free and interacting fields the same, the lagrangian would be a Lorentz-scalar density, and the theory would be Lorentz covariant. Thus it must be concluded that, for these theories, which are acausal in the sense of Velo and Zwanziger, the free and interacting fields have different Lorentz-transformation properties.

That the acausal theories, presented here, are not Lorentz covariant, has been seen to be manifested in the Cauchy problem, for the hyperbolic system plus constraints, which are equivalent to the original field equations, not taking the same form in all inertial frames. There exists an inertial frame in which the constraints do not determine all the dependent field components.

As was noted in § 2 for the spin-1 case, and as may be seen for the spin-$\frac{3}{2}$ case by comparing (15) and the expression for the characteristic determinant in Singh (1973), the quantity, which determines whether or not the constraints are always solvable for the dependent field components, is essentially the characteristic determinant, and this also determines the causal nature of the propagation of the fields (Velo and Zwanziger 1969a, b, 1971). Thus the results of the present paper are expected to be more general, and it is conjectured that classical theories, based on relativistic wave equations for unique mass and spin, are causal if and only if they are Lorentz covariant. If this conjecture were true, the calculation of the characteristic determinant would provide a method of determining whether or not such a field theory is Lorentz covariant, which is much simpler than the direct approach of constructing the generators of the Lorentz group, and their Poisson-bracket relations. The new role, found here, for the characteristic determinant, emphasizes its already known importance for both classical and quantum relativistic field theories (Velo and Zwanziger 1969a, b, 1971, Lee and Yang 1962, Jenkins 1973b, 1974).

Finally, some remarks are made on the situation of interacting massive spin-$\frac{3}{2}$ theories. As seen in § 3, theories bilinear in the spin-$\frac{3}{2}$ and Dirac fields are, with the exception of a trivial case, all lacking Lorentz covariance. Since such theories may be expected to be less pathological than theories involving higher degree polynomials of the spin-$\frac{3}{2}$ field the situation is acute, and is not expected to be any better for higher spins.

At present, there seem to be two directions which could lead out of this impasse. Both involve demanding that higher-spin fields fall into families, there being relations between the masses and/or couplings of the members of each family.

Firstly, Velo (1973) has suggested that there is a small chance that demanding higher-spin fields fall into multiplets of given spin, there being relations between the couplings within each multiplet, may lead to causal (and Lorentz-covariant) theories. This idea arose from his discussion of a generalization of the massive Yang–Mills field (Yang and Mills 1954).

Secondly, by considering fields which describe several masses and spins, there being relations between the masses and the couplings, problems with constraints and acausality can be avoided. See, for example, the work of Nagpal (1973) on the Bhabha fields. A logical extension of this approach would be to consider infinite-component wave equations describing Regge families.

It is concluded that the search for Lorentz-covariant higher-spin theories seems to guide one in the directions of multi-mass and multi-spin (possibly infinite-component) wave equations, and symmetries.

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